

On Machin's formula with Powers of the Golden Section

FLORIAN LUCA

Instituto de Matemáticas

Universidad Nacional Autónoma de México

C.P. 58089, Morelia, Michoacán, México

`fluca@matmor.unam.mx`

PANTELIMON STĂNICĂ

Naval Postgraduate School

Department of Applied Mathematics

Monterey, CA 93943, USA

`pstanica@nps.edu`

April 11, 2008

Abstract

In this note, we find all solutions of the equation $\frac{\pi}{4} = a \arctan(\phi^\kappa) + b \arctan(\phi^\ell)$, in integers κ and ℓ and rational numbers a and b , where ϕ is the golden section.

MSC: 11D45, 11D85, 11R04, 11R29.

Keywords: π , golden section, primitive divisors, Fibonacci, Lucas numbers.

Motivated by Machin's formula (see [Borwein and Bailey 03, p. 105])

$$\frac{\pi}{4} = 4 \arctan(5^{-1}) - \arctan(239^{-1}),$$

several researchers (see, for instance, [Séroul 00] and the references therein) generalized it to identities of the form

$$\frac{k\pi}{4} = m \arctan(u^{-1}) + n \arctan(v^{-1}),$$

Report Documentation Page				Form Approved OMB No. 0704-0188	
Public reporting burden for the collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden, to Washington Headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington VA 22202-4302. Respondents should be aware that notwithstanding any other provision of law, no person shall be subject to a penalty for failing to comply with a collection of information if it does not display a currently valid OMB control number.					
1. REPORT DATE 11 APR 2008		2. REPORT TYPE		3. DATES COVERED 00-00-2008 to 00-00-2008	
4. TITLE AND SUBTITLE On Machin's formula with Powers of the Golden Section				5a. CONTRACT NUMBER	
				5b. GRANT NUMBER	
				5c. PROGRAM ELEMENT NUMBER	
6. AUTHOR(S)				5d. PROJECT NUMBER	
				5e. TASK NUMBER	
				5f. WORK UNIT NUMBER	
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) Naval Postgraduate School, Department of Applied Mathematics, Monterey, CA, 93943				8. PERFORMING ORGANIZATION REPORT NUMBER	
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)				10. SPONSOR/MONITOR'S ACRONYM(S)	
				11. SPONSOR/MONITOR'S REPORT NUMBER(S)	
12. DISTRIBUTION/AVAILABILITY STATEMENT Approved for public release; distribution unlimited					
13. SUPPLEMENTARY NOTES International Journal of Number Theory vol.5:6 (2009), pp. 973-979.					
14. ABSTRACT In this note, we find all solutions of the equation $4 = a \arctan(\quad) + b \arctan(\quad)$; in integers a and b and rational numbers a and b, where ϕ is the golden section.					
15. SUBJECT TERMS					
16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT Same as Report (SAR)	18. NUMBER OF PAGES 8	19a. NAME OF RESPONSIBLE PERSON
a. REPORT unclassified	b. ABSTRACT unclassified	c. THIS PAGE unclassified			

where u, v, k are positive integers and m, n are nonnegative integers. Such formulae are useful in the computation of π . It is completely natural to ask whether such formulas will hold if u, v are replaced by numbers from a larger class than reciprocals of integers (like rational, algebraic, etc.). Chan, and Chan and Ebbing (see [Chan 07, Chan and Ebbing 06]) investigated analogues of Machin's formula with some rational coefficients when the rational numbers 5^{-1} and 239^{-1} are replaced by small powers of negative exponent of the golden section $\phi = (1 + \sqrt{5})/2$ and they found three such formulas, where the pairs of exponents are $(\kappa, \ell) = (3, 1), (5, 3), (6, 2)$. In this short note, we show that up to some trivial transformations, there are no others besides the previous ones.

Theorem 1. *If $\kappa, \ell \in \mathbb{Z} \setminus \{0\}$ with $|\kappa| \geq |\ell|$, $\kappa + \ell \neq 0$ and $a, b \in \mathbb{Q}$ such that*

$$\frac{\pi}{4} = a \arctan(\phi^\kappa) + b \arctan(\phi^\ell), \quad (1)$$

then

$$(a, b, \kappa, \ell) \in \left\{ \begin{array}{l} \left(\frac{1}{3}, \frac{1}{3}, 3, 1\right), (1, 1, -3, -1), (-1, 1, -3, 1), (1, -1, 3, -1), \\ \left(\frac{1}{5}, \frac{2}{5}, 6, 2\right), (1, 2, -6, -2), \left(-\frac{1}{3}, \frac{2}{3}, -6, 2\right), (1, -2, 6, -2), \\ \left(\frac{1}{7}, \frac{3}{7}, 5, 3\right), (1, 3, -5, -3), \left(-\frac{1}{5}, \frac{3}{5}, -5, 3\right), (1, -3, 5, -3) \end{array} \right\}.$$

Before proving the theorem, we start with a few comments about the proof, which relies on an identity of [Borwein and Borwein 87] and known facts on algebraic numbers (see [Washington 97]). When $|\kappa| > |\ell|$, we embed identity (1) in the biquadratic class number 1 field $\mathbb{K} = \mathbb{Q}[\phi, i]$, and use some known results on the prime factors of Fibonacci and Lucas sequences, to show nonexistence of solutions of (1) besides the mentioned ones. If $\kappa = \ell$, we show that the mentioned identity is equivalent to an equation in $\mathbb{Q}[\zeta_{20}]$, which has no solutions. We exclude the cases $\kappa + \ell = 0$ since it is well-known that if x is a positive real number, then

$$\frac{\pi}{2} = \arctan x + \arctan \left(\frac{1}{x} \right). \quad (2)$$

We also note that whenever (a, b, κ, ℓ) is a solution of equation (1) with $a \neq 1/2$, then using the fact that $\arctan(\phi^{-\kappa}) + \arctan(\phi^\kappa) = \pi/2$, one gets that $(-a/(1 - 2a), b/(1 - 2a), -\kappa, \ell)$ is also a solution of equation (1).

We mention that Machin-like formulas with powers of other irrationals exist in the literature, an example being

$$\frac{\pi}{2} = 2 \arctan \left(\frac{1}{\sqrt{2}} \right) + \arctan \left(\frac{1}{\sqrt{8}} \right).$$

Proof. Let $a = u/w$, $b = v/w$, where $u, v, w > 0$ are integers with w and $\gcd(u, v)$ coprime. Then the given relation becomes

$$\frac{w\pi}{4} = u \arctan(\phi^\kappa) + v \arctan(\phi^\ell). \quad (3)$$

By the result on page 345 in [Borwein and Borwein 87], the above relation holds if and only if

$$(1 - i)^w(\phi^{-\kappa} + i)^u(\phi^{-\ell} + i)^v$$

is real, which is equivalent to the fact that

$$(1 - i)^w(1 + i\phi^\kappa)^u(1 + i\phi^\ell)^v = (1 + i)^w(1 - i\phi^\kappa)^u(1 - i\phi^\ell)^v.$$

Raising the above equation to the fourth power and using the fact that $(1 + i)^4 = (1 - i)^4 = -4$, we get that

$$(1 + i\phi^\kappa)^{4u}(1 + i\phi^\ell)^{4v} = (1 - i\phi^\kappa)^{4u}(1 - i\phi^\ell)^{4v}. \quad (4)$$

If $\kappa = \ell$, we then get that

$$\left(\frac{1 + i\phi^\kappa}{1 - i\phi^\kappa}\right)^{4(u+v)} = 1,$$

which implies either that $u + v = 0$ or that $\zeta = (1 + i\phi^\kappa)/(1 - i\phi^\kappa)$ is a root of 1. If $u + v = 0$, then relation (3) leads to $w\pi/4 = 0$, which is impossible. Thus, ζ is a root of unity. For a positive integer m we write $\zeta_m = \exp(2\pi i/m)$. Since $\phi \in \mathbb{Q}[\zeta_5]$ and $i \in \mathbb{Q}[\zeta_4]$, we get that $\zeta \in \mathbb{Q}[\phi, i] \subseteq \mathbb{Q}[\zeta_5, \zeta_4] = \mathbb{Q}[\zeta_{20}]$. Hence, $\zeta = \zeta_{20}^n$ for some $n \in \{0, \dots, 19\}$. This implies that

$$\phi^\kappa = -i \left(\frac{\zeta_{20}^n - 1}{\zeta_{20}^n + 1} \right) = -\tan\left(\frac{n\pi}{20}\right).$$

One can now check that the above equation has no solution with $n \in \{0, \dots, 19\}$ and $\kappa \in \mathbb{Z} \setminus \{0\}$.

Assume now that $|\kappa| > |\ell|$. Let $\mathbb{K} = \mathbb{Q}[\phi, i]$. It is known that \mathbb{K} is biquadratic and has class number 1 (see [Hideo 86]). We shall show that $|\kappa| \leq 12$. Let π be any prime ideal dividing $1 + i\phi^\kappa$. From relation (4), we get that π either divides $1 - i\phi^\kappa$, or it divides $1 - i\phi^\ell$. If π divides $1 - i\phi^\kappa$, it follows that π divides 2. Hence, if π does not divide 2, then π divides both $1 + i\phi^\kappa$ and $1 - i\phi^\ell$. Note that

$$N_{\mathbb{K}}(1 + i\phi^\kappa) = (1 + \phi^{2\kappa})(1 + (-\phi)^{-2\kappa}) = (\phi^\kappa + \phi^{-\kappa})^2.$$

Hence, $N_{\mathbb{K}}(1 + i\phi^\kappa)$ is either $5F_\kappa^2$ or L_κ^2 according to whether κ is odd or even. Here, F_m and L_m are the regular Fibonacci and Lucas numbers given by $F_0 = 0$, $F_1 = 1$, $L_0 = 2$, $L_1 = 1$ and $F_{m+2} = F_{m+1} + F_m$, $L_{m+2} = L_{m+1} + L_m$ for all $m \geq 0$. Similarly, $N_{\mathbb{K}}(1 + i\phi^\ell)$ is either $5F_\ell^2$ or L_ℓ^2 according to whether ℓ is odd or even.

Assume now that $|\kappa| \geq 13$. First, assume that κ is odd. Then there exists a prime number p dividing F_κ which is primitive; i.e., such that p does not divide F_μ for any positive integer $\mu < \kappa$ (see [Bilu et al. 01] for more on primitive divisors). Let π be any prime ideal in \mathbb{K} dividing p . If π divides $1 + i\phi^\kappa$, then π divides either 2 (which is impossible because $p > 2$), or π divides $1 - i\phi^\ell$, which divides either $5F_\ell$ or L_ℓ according to whether ℓ is odd or even. Hence, p divides either $5F_\ell$ or L_ℓ . Since p is primitive for F_κ , p cannot divide $5F_\ell$. If $p \mid L_\ell$, then since $L_\ell \mid F_{2\ell}$, we then get that $\kappa \mid 2\ell$, and since κ is odd we get that $\kappa \mid \ell$, which is impossible. It remains to show that we can always assume that p is divisible by some prime in \mathbb{K} dividing $1 + i\phi^\kappa$. Indeed, let π be some prime divisor of p . If $\pi \mid 1 + i\phi^\kappa$, then we are done. If $\pi \mid 1 - i\phi^\kappa$, then the complex conjugate of π (which also divides p) must divide $1 + i\phi^\kappa$. Finally, if π divides $1 \pm i(-\phi)^{-\kappa}$, then the image of π via the Galois automorphism of \mathbb{K} which sends $-\phi^{-1}$ to ϕ and fixes i will send π into a prime ideal (still dividing p) divisor of $1 \pm i\phi^\kappa$, which is a situation already treated. This takes care of the proof of the fact that $|\kappa| \leq 12$ if κ is odd. If κ is even, then the same argument using the existence of primitive divisors for the Lucas sequence shows that $|\kappa| \leq 12$ also. It remains to compute the examples. Since the remaining of our analysis is based on the arithmetic structure of F_m and L_m for $m = \kappa, \ell$ and since $F_{-m} = \pm F_m$ and $L_{-m} = L_m$, we assume that $0 < \ell < \kappa$ (or, we replace κ and ℓ by their absolute values).

Let $\pi \nmid 2$ be a prime ideal in \mathbb{K} such that

$$\pi \mid 1 + i\phi^\kappa \quad \text{and} \quad \pi \mid 1 - i\phi^\ell. \quad (5)$$

Clearly, π cannot divide any power of ϕ because ϕ is a unit. Further, $\pi \mid (1 + i\phi^\kappa) - (1 + i\phi^\ell) = i\phi^\ell(1 + \phi^{\kappa-\ell})$, and since $\pi \nmid i\phi^\ell$, we get that

$$\pi \mid 1 + \phi^{\kappa-\ell} \quad \text{and} \quad \pi \mid 1 - i\phi^\ell. \quad (6)$$

Furthermore, $\pi \mid \phi^{\kappa-\ell} + i\phi^\ell = i\phi^\ell(1 - i\phi^{\kappa-2\ell})$, which implies that

$$\pi \mid 1 - i\phi^{\kappa-2\ell} \quad \text{and} \quad \pi \mid 1 - i\phi^\ell. \quad (7)$$

Continuing in this manner, we obtain that $\pi \mid -i\phi^{\kappa-2\ell} + i\phi^\ell = i\phi^\ell(1 - \phi^{\kappa-3\ell})$, which implies that

$$\pi \mid 1 - \phi^{\kappa-3\ell} \quad \text{and} \quad \pi \mid 1 - i\phi^\ell. \quad (8)$$

Now let $\kappa = r\ell + t$, where $0 \leq t < \ell$. Fix $\kappa \leq 12$, and let $\ell \neq 0$ to be the least positive integer satisfying (5). The previous analysis suggests considering the following cases.

Case 1. $r \equiv 0 \pmod{4}$.

From (5) and the previous analysis, we get that

$$\pi \mid 1 + i\phi^t \quad \text{and} \quad \pi \mid 1 - i\phi^\ell,$$

where $t < \ell$, which implies that $\pi \mid i\phi^t(1 + \phi^{\ell-t})$. Thus, $\pi \mid 1 + \phi^{\ell-t}$ and $\pi \mid 1 + i\phi^t$, which in turn leads to

$$\pi \mid \phi^{\ell-t} - i\phi^t.$$

If on the one hand $t < \ell \leq 2t < 2\ell$, then $\pi \mid \phi^{\ell-t}(1 - i\phi^{2t-\ell})$; that is, $\pi \mid 1 - i\phi^{2t-\ell}$ with $2t - \ell < \ell$, which is in contradiction with the assumed minimality of ℓ . If on the other hand $2t \leq \ell$, we then get $\pi \mid -i\phi^t(1 - i\phi^{\ell-2t})$. Thus, $\pi \mid 1 - i\phi^{\ell-2t}$, which again either contradicts the minimality of ℓ (if $2t < \ell$), or the fact that $\pi \nmid 2$ (if $2t = \ell$).

Case 2. $r \equiv 1 \pmod{4}$.

From (6), we get that

$$\pi \mid 1 + \phi^t \quad \text{and} \quad \pi \mid 1 - i\phi^\ell,$$

which implies $\pi \mid \phi^t(1 + i\phi^{\ell-t})$. So,

$$\pi \mid 1 + i\phi^{\ell-t} \quad \text{and} \quad \pi \mid 1 + \phi^t.$$

If $\ell \geq 2t$, then from the previous relations we get that $\pi \mid 1 - i\phi^{\ell-2t}$, which is a contradiction. If $\ell < 2t$, then $\pi \mid i\phi^{\ell-t} - \phi^t = i\phi^{\ell-t}(1 + i\phi^{2t-\ell})$, and so, $\pi \mid 1 + i\phi^{2t-\ell}$. Then, since $1 \leq \ell < \kappa \leq 12$, we have $r \in \{1, 5, 9\}$ and a simple computation reveals the possibilities

$$(\kappa, \ell) \in \{(5, 3), (7, 4), (8, 5), (9, 5), (10, 6), (11, 6), (11, 7), (12, 7)\}. \quad (9)$$

Case 3. $r \equiv 2 \pmod{4}$.

From (7), we get that

$$\pi \mid 1 - i\phi^t \quad \text{and} \quad \pi \mid 1 - i\phi^\ell$$

with $t < \ell$, which contradicts again the minimality of ℓ .

Case 4. $r \equiv 3 \pmod{4}$.

From (8), we get

$$\pi \mid 1 - \phi^t \quad \text{and} \quad \pi \mid 1 - i\phi^\ell.$$

If on the one hand $t \neq 0$, then $\pi \mid -\phi^t + i\phi^\ell = -\phi^t(1 - i\phi^{\ell-t})$, which implies $\pi \mid 1 - i\phi^{\ell-t}$, in contradiction with the minimality of ℓ . If on the other hand $t = 0$, then $\kappa = r\ell$, with $r \in \{3, 7\}$, and we obtain the possibilities

$$(\kappa, \ell) \in \{(3, 1), (6, 2), (7, 1), (9, 3), (12, 4)\}. \quad (10)$$

The authors of [Chan and Ebbing 06] found the Machin-like formulas with powers (κ, ℓ) of the golden section, where $(\kappa, \ell) = (3, 1), (5, 3), (6, 2)$; that is,

$$\begin{aligned} \frac{3\pi}{4} &= \arctan(\phi) + \arctan(\phi^3), \\ \frac{5\pi}{4} &= 2\arctan(\phi^2) + \arctan(\phi^6), \\ \frac{7\pi}{4} &= 3\arctan(\phi^3) + \arctan(\phi^5). \end{aligned}$$

To get the remaining examples listed in the statement of Theorem 1, we note that whenever (a, b, κ, ℓ) is a solution of equation (1) with $a \neq 1/2$, then using the fact that $\arctan(\phi^{-\kappa}) + \arctan(\phi^\kappa) = \pi/2$, one gets that $(-a/(1 - 2a), b/(1 - 2a), -\kappa, \ell)$ is also a solution of equation (1). The remaining nine solutions of equation (1) are all obtained in the above fashion from the above three solutions with $(\kappa, \ell) = (3, 1), (5, 3), (6, 2)$.

We now need to deal with the other pairs in (9) and (10), namely

$$\begin{aligned} (\kappa, \ell) \in \{(7, 1), (7, 4), (8, 5), (9, 3), (9, 5), \\ (10, 6), (11, 6), (11, 7), (12, 4), (12, 7)\}. \end{aligned} \quad (11)$$

Assume, say that $(\kappa, \ell) = (7, 1)$. Take a prime ideal π in \mathbb{K} such that π divides both $1 + i\phi^7$ and 13 (note that $F_7 = 13$). Such a prime ideal divides neither 2, nor $1 - i\phi$, since otherwise the rational prime 13 would divide $N_{\mathbb{K}}(1 - i\phi) = (\phi + \phi^{-1})^2 = 5$, and so we get a contradiction. Similarly, we can remove all the remaining possibilities from (11) since in each instance there is a rational prime divisor of $5F_{\kappa}^2$ (with κ odd), or L_{κ}^2 (for κ even) which is not a rational prime divisor of the corresponding norm of $1 - i\phi^{\ell}$.

We are not yet done, since so far we have merely shown that if (a, b, κ, ℓ) with $|\kappa| \geq |\ell|$ and $\kappa + \ell \neq 0$ satisfies equation (1), then $(|\kappa|, |\ell|) = (3, 1), (5, 3), (6, 2)$. In order to finish, we need to show that each such solution (a, b, κ, ℓ) is uniquely determined by its last two components. Assume that this is not so. Then there exists a pair (κ, ℓ) such that (a, b, κ, ℓ) and (a', b', κ, ℓ) are both solutions of equation (1) for two distinct pairs (a, b) and (a', b') of rational numbers. It then follows that $\arctan(\phi^{\ell})$ and π are linearly dependent over the rationals. Thus, there exists a rational number r such that $\phi^{\ell} = \tan(r\pi)$. By replacing r with $1/2 - r$, we may assume that $\ell > 0$. Then $(2\cos(r\pi))^2 = 4/(1 + \phi^{2\ell})$. However, $2\cos(r\pi) = e^{ir\pi} + e^{-ir\pi}$ is an algebraic integer. Thus, $4/(1 + \phi^{2\ell})$ is an algebraic integer. When $\ell = 1$ and 2 this last number takes the values

$$\frac{2(\sqrt{5} - 1)}{\sqrt{5}} \quad \text{and} \quad \frac{2(3 - \sqrt{5})}{3},$$

and none of them is an algebraic integer. Since $\frac{4}{1 + \phi^2} = (1 - \phi^2 + \phi^4) \cdot \frac{4}{1 + \phi^6}$, we get that if the number $4/(1 + \phi^{2\ell})$ is an algebraic integer when $\ell = 3$, then it is also for $\ell = 1$, and we have just seen that this is impossible. This indeed completes the proof of Theorem 1. \square

Acknowledgements. The authors would like to thank the anonymous referee for comments which improved the quality of this paper. Work by the first author was done while he visited Williams College. He was also supported in part by Grant SEP-CONACYT 46755. The second author was partially supported by the Naval Postgraduate School RIP funding.

References

[Bilu et al. 01] Y. Bilu, G. Hanrot, P.M. Voutier, ‘Existence of primitive divisors of Lucas and Lehmer numbers (with an appendix by M.

- Mignotte)', *J. Reine Angew. Math.* **539** (2001), 75-122.
- [Borwein and Bailey 03] J. M. Borwein and D. Bailey, *Mathematics by Experiment: Plausible Reasoning in the 21st Century*, Natick, MA: A. K. Peters, 2003.
- [Borwein and Borwein 87] J. M. Borwein and P. B. Borwein, *Pi & the AGM: A Study in Analytic Number Theory and Computational Complexity*, New York: Wiley, 1987.
- [Chan 07] H.-C. Chan, 'Machin-type formulas expressing π in terms of ϕ ', *Preprint*, 2007.
- [Chan and Ebbing 06] H.-C. Chan and S. Ebbing, ' π in terms of ϕ : Some recent developments', in *Proceedings of the Twelfth International Conference in Fibonacci Numbers*, to appear.
- [Hideo 86] Y. Hideo, 'Imaginary Bicyclic Biquadratic Fields With The Real Quadratic Subfield Of Class-Number One', *Nagoya Math. J.* **102** (1986), 91–100.
- [Sérroul 00] R. Sérroul, 'Machin Formulas', §9.3 in *Programming for Mathematicians*, Berlin: Springer-Verlag, 240–252, 2000.
- [Washington 97] L. C. Washington, *Introduction to Cyclotomic Fields* (2nd ed.), Graduate Texts in Mathematics 83, Springer-Verlag, New York, 1997.